COMPLETELY MONOTONE SEQUENCES AS INVARIANT MEASURES(1)

BY
RICHARD SCOVILLE

Introduction. A completely monotone sequence $\boldsymbol{a} = (a_0, a_1, a_2, \cdots)$ of real numbers is a sequence which satisfies the conditions

$$(1) \qquad (\Delta^n \boldsymbol{a})_k \geqq 0 \quad (n = 0,1,2,\cdots; \quad k = 0,1,2,\cdots)$$

where the operator Δ, acting on any sequence $\boldsymbol{a}$, is the sequence $\Delta\boldsymbol{a}$ defined by

$$(\Delta \boldsymbol{a})_k = a_k - a_{k+1} \quad (k = 0,1,2,\cdots).$$

During the first part of the paper we will suppose that the completely monotone sequence $\boldsymbol{a}$ also satisfies

$$(2) \qquad \sum_{n=0}^{\infty} a_n = 1.$$

Using the sequence $\boldsymbol{a}$, we will construct a probability space $(X, \mathscr{B}, \mu)$ and a measure-preserving transformation T on it. This transformation will not, in general, be ergodic. We then make use of the Kryloff-Bogoliouboff decomposition [4] (as extended by Oxtoby [5]) of a measure into its ergodic parts. Writing μ as an integral of measures for which T *is* ergodic, and observing the effect of this decomposition on the sequence $\boldsymbol{a}$, we obtain the classical reduction of $\boldsymbol{a}$ to the form

$$a_k = \int_0^1 t^k dF(t) \quad (k = 0,1,2,\cdots).$$

An inversion formula, similar to Feller's [2], giving F in terms of $\boldsymbol{a}$ can also be obtained in this way. In the last part of the paper we investigate the transformation T itself, and leave several interesting questions unanswered.

This paper is the major part of the author's dissertation prepared at Yale University under the direction of S. Kakutani, whom I wish to thank for his advice, encouragement and great skill.

$(X, \mathscr{B}, \mu)$ and the transformation T. Let us assume, then, that a sequence $\boldsymbol{a}$ satisfying (1) and (2) has been given and proceed with the construction of $(X, \mathscr{B}, \mu)$.

Received by the editors April 15, 1963.

(1) This research was supported in part by the United States Air Force under Grant AF AFOSR 61-51, monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

Let I_∞ be the one-point compactification of the non-negative integers, with ∞ the added point. We form the compact space

$$X'' = \prod_{i=1}^{\infty} X_i \quad (X_i = I_\infty, i = 1,2,3,\cdots)$$

and consider the closed subset X' of X'' consisting of all sequences in X'' which are increasing (not necessarily strictly increasing), thinking of ∞ as being greater than every integer.

By a rectangle $(r_1, r_2, r_3, \cdots, r_k)$ where $0 \leqq r_1 \leqq r_2 \leqq \cdots \leqq r_k$, is meant all sequences in X' beginning with $r_1, r_2, r_3, \cdots, r_k$, i.e., all sequences ω in X' for which $\omega_i = r_i$ $(i = 1,2,3,\cdots,k)$. The measure of a rectangle $R = (r_1, r_2, \cdots, r_k)$ is defined as follows:

$$\mu(R) = \begin{cases} (\Delta^{k-1}\boldsymbol{a})_{r_k} & \text{if} \quad r_k \neq \infty, \\ 0 & \text{if} \quad r_k = \infty. \end{cases} \tag{3}$$

Let $\mathscr{A}$ be the (finitely additive) algebra consisting of all sets of X' which depend on only a finite number of coordinates, i.e., $\mathscr{A} = \bigcup \mathscr{A}_n$ where $\mathscr{A}_n$ consists of sets which can be written as the union of rectangles of length n. The measure of any set in $\mathscr{A}$ may now be defined by writing the set as a union of disjoint rectangles. This measure can be extended to a countably additive measure (again called μ) defined on $\mathscr{B}$, the smallest σ-field containing $\mathscr{A}$. Since μ is a regular measure on $\mathscr{A}$, this can be accomplished, for instance, by means of a theorem of Alexandroff [2, p. 138].

We let X be the subset of X' consisting of those sequences in which ∞ does not appear. The complement of X in X' may be written

$$X^c = \bigcup_{n=1}^{\infty} \bigcup_{i_1 \leqq i_2 \leqq \cdots \leqq i_n}^{\infty} (i_1, i_2, \cdots, i_n, \infty).$$

This set clearly has measure zero, no matter what the sequence $\boldsymbol{a}$ may be, so we see that X has measure one. From now on when we speak of rectangles we mean rectangles restricted to the space X.

The transformation T is defined at a point $\omega = (\omega_1, \omega_2, \omega_3, \cdots)$ of X as follows:

$$T\omega = \begin{cases} (\underbrace{0,0,\cdots,0,}_{n-1} \quad \omega_n + 1, \quad \omega_{n+1}, \cdots) & \text{if } \omega_1 = \omega_n < \omega_{n+1}, \\ (0,0,0,\cdots) & \text{if } \omega_1 = \omega_2 = \omega_3 = \cdots. \end{cases} \tag{4}$$

Suppose $R = (r_1, r_2, \cdots, r_n)$ is a rectangle in X. If $r_1 \neq 0$, the image of R under T^{-1} is again a rectangle: $(r_1 - 1, r_2, \cdots, r_n)$. If $0 = r_1 = r_2 = \cdots = r_k$ but $r_{k+1} \neq 0$ where $k < n$, the inverse image of R is the rectangle

$$(\underbrace{r_{k+1} - 1, r_{k+1} - 1, \cdots, r_{k+1} - 1,}_{k+1} \quad r_{k+2}, \cdots, r_n).$$

If, finally, $0 = r_1 = r_2 = \cdots = r_n$, the inverse image of R is, except for the constant sequences, the disjoint union

$$\bigcup_{p=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcup_{i=0}^{j-1} (\overbrace{i, i, \cdots, i}^{n+p}, j).$$

Hence in any case the inverse image under T of any rectangle R is essentially a union of rectangles and so in particular is measurable. It is possible to calculate the measure of $T^{-1}R$ directly frcm formula (3) to show that T is measure-preserving, but the following "geometric" description of T is more enlightening but not essential to the development of the paper. This idea has been used before, for instance in [1].

We construct a sequence of line segments, $B_0, B_1, B_2, \cdots$, one above the other as in Figure la so that the segment B_n has length a_n. Let us call the object so obtained a *building* and also regard it as a measure space by giving to each linear Borel set in the building its ordinary (linear) Lebesgue measure. Now if there were a measure-preserving transformation, say S, defined on B_0, we could construct a measure-preserving transformation T on the building B by sending any point to the point directly above on the next floor, if possible, otherwise by descending straight down to B_0 and moving by S. Then S is simply the transformation *induced* by T on B_0, of which we will speak later. What remains then is to define the transformation S on B_0.

We can partially define S on B_0 by writing B_0 itself as a building as in Figure 1b so that the segment B_n' has length $a_n - a_{n+1}$. S is now partially defined on B_0 simply by rising one floor, if possible. To continue its definition, we must write B_0' as a building in a way similar to the way we wrote B_0 as a building. Eventually then, S and hence T will be completely defined except perhaps on a set of measure zero. That we continue to get a building at each stage is due to the requirement (1) on the sequence $\boldsymbol{a}$. The floors of the nth stage building will have lengths $(\Delta^{n-1}\boldsymbol{a})_0, (\Delta^{n-1}\boldsymbol{a})_1, \cdots$.

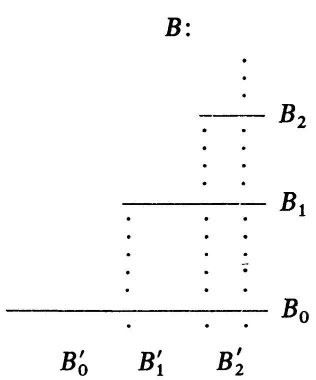

Figure 1a

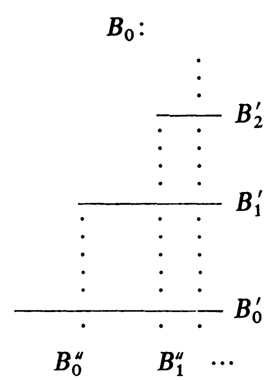

Figure 1b

The transformation T defined on X is said to be *conjugate* to the transformation S on Y if there is a 1-1 measure-preserving map ϕ from (almost all of) X onto (almost all of) Y such that ϕ^{-1} is measurable and $T = \phi^{-1} \circ S \circ \phi$. We can now show the relation between the two measure-preserving transformations defined so far.

LEMMA. *The transformation T defined on B is conjugate to the transformation T defined on X by* (4).

Proof. We can give coordinates to a point ω in B. Let ω_1 be the integer satisfying $\omega \in B_{\omega_1}$. Drop ω to the first floor of B and call this point ω'. Let ω_2 be the integer satisfying $\omega' \in B'_{\omega_2}$. Drop ω' to the first floor of B_0 and call this point ω''. Let ω_3 satisfy $\omega'' \in B''_{\omega_3}$, etc. This correspondence $\omega \to (\omega_1, \omega_2, \cdots)$ from B to X is easily seen to establish the conjugacy of T on B and T on X.

The measures μ for which T is ergodic. Since we will have to speak of several completely monotone sequences and their corresponding measures, we now write μ_a for the measure defined on X by means of the sequence $\boldsymbol{a}$, and T_a for the corresponding transformation previously called simply T. Of course all the transformations T_a are identical, but we may wish to distinguish a particular measure μ_a which T preserves. In this case, we write T_a. For instance we could ask are T_a and T_b conjugate?

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are completely monotone sequences which also satisfy (2), clearly any convex combination $\alpha\boldsymbol{a} + \beta\boldsymbol{b}$ is another. In fact, because of the linearity of Δ,

$$\alpha\mu_a + \beta\mu_b = \mu_{\alpha a + \beta b}. \tag{5}$$

Also, given a completely monotone sequence $\boldsymbol{a} \neq (1,0,0,0,\cdots)$, we can form two others, namely $\sigma\boldsymbol{a}$ and $\tau\boldsymbol{a}$ defined by

$$\begin{aligned} (\sigma\boldsymbol{a})_n &= a_{n+1}/(1-a_0), \\ (\tau\boldsymbol{a})_n &= (a_n - a_{n+1})/a_0 \qquad (n = 0,1,2,\cdots). \end{aligned} \tag{6}$$

These sequences also satisfy (2) if $\boldsymbol{a}$ does.

Let $T_a \mid C$ be the transformation induced by T_a on the subset C of X of all sequences in X having first coordinate zero, and let $T_a \mid \tilde{C}$ be the induced transformation on $\tilde{C} = X - C$. Hence for instance $T_a \mid C$ is defined at a point ω of C by $T_a \mid C(\omega) = T^k\omega$ where k is the first positive integer for which $T^k(\omega) \in C$. $T_a \mid C$ is a measure preserving transformation on C. We call the measure ρ defined on subsets E of C by $\rho(E) = \mu_a(E)/\mu_a(C)$ the *induced* measure on C.

LEMMA. *Suppose $\boldsymbol{a} \neq (1,0,0,\cdots)$. Then the transformation $T_a \mid C$ on C with its induced measure, is conjugate to the transformation $T_{\tau a}$ on X. Also, $T_a \mid \tilde{C}$ on $\tilde{C}$ with its induced measure is conjugate to $T_{\sigma a}$.*

Proof. A glance at the buildings is probably enough to convince the reader of this fact, but we will construct mappings $\phi: C \to X$ and $\psi: \tilde{C} \to X$ which give the conjugacy explicitly. Let $\phi(\omega) = (\omega_2, \omega_3, \cdots)$, $\omega \in C$ and $\psi(\omega) = (\omega_1 - 1, \omega_2 - 1, \cdots)$, $\omega \in \tilde{C}$. Suppose $R = (r_1, r_2, \cdots, r_k)$ is a rectangle in X. We must show that

$$\mu_{\tau a}(R) = \mu_a(\phi^{-1} R) / \mu_a(C)$$

and

$$\mu_{\sigma a}(R) = \mu_a(\psi^{-1} R) / \mu_a(\tilde{C}).$$

Since $\phi^{-1}(R) = (0, r_1, r_2, \cdots, r_k)$, from (3) and (6) we get

$$\mu_{\tau a}(R) = (\Delta^{k-1} \tau \boldsymbol{a})_{r_k} = (1 / a_0)\, (\Delta^k \boldsymbol{a})_{r_k}$$

$$= (1 / \mu_a(C)) \mu_a(\phi^{-1} R).$$

Similarly, $\psi^{-1}(R) = (r_1 + 1, r_2 + 1, \cdots, r_k + 1)$ so that

$$\mu_{\sigma a}(R) = (\Delta^{k-1} \sigma \boldsymbol{a})_{r_k} = (1 / (1 - a_0))(\Delta^{k-1} \boldsymbol{a})_{r_k + 1} = (1 / \mu_a(\tilde{C})) \mu_a(\psi^{-1}(R)).$$

These equations show that ϕ and ψ are indeed measure-preserving isomorphisms between C with its induced measure and X, and between $\tilde{C}$ with its induced measure and X, resp. If $\omega = (0, \omega_2, \omega_3, \cdots)$ is any point in C, we see that $\phi \circ T_a \mid C(\omega) = T(\omega_2, \omega_3, \cdots) = T \circ \phi(\omega) = T_{\tau a} \circ \phi(\omega)$ and in the same way, we can show that $\psi \circ T_a \mid \tilde{C} = T_{\sigma a} \circ \psi$, proving the lemma.

LEMMA. *If T_a is ergodic, then $\boldsymbol{a}$ is of the form*

$$a_n = \theta^n (1 - \theta) \quad (n = 0, 1, 2, \cdots) \tag{7}$$

for some θ, $0 \leqq \theta < 1$. (We set $\theta^0 \equiv 1$).

Proof. Suppose T_a is ergodic. If $\boldsymbol{a} = (1, 0, 0, \cdots)$ then certainly $\boldsymbol{a}$ has the form required in the lemma. We exclude this case from what follows. Hence the induced transformations $T_a \mid C$ and $T_a \mid \tilde{C}$ are defined, and they also are ergodic, since any transformation induced by an ergodic transformation is itself ergodic. Now by the previous lemma these induced transformations are conjugate to $T_{\tau a}$ and $T_{\sigma a}$ respectively so that $T_{\tau a}$ and $T_{\sigma a}$ are ergodic. Or what comes to the same thing T is ergodic with respect to the three measures $\mu_a, \mu_{\tau a}$ and $\mu_{\sigma a}$. But by (5) $\mu_a = (1 - a_0) \mu_{\sigma a} + a_0 \mu_{\tau a}$. This implies, as is well known, that $\mu_a = \mu_{\tau a} = \mu_{\sigma a}$. In particular, $a_n = a_{n+1} / (1 - a_0)$, so that $a_n = (1 - a_0)^n a_0$ $(n = 0, 1, 2, \cdots)$. Setting $\theta = 1 - a_0$, we get the lemma. The measure μ_a, where $\boldsymbol{a}$ is given by (7), will sometimes be written μ_θ.

We conclude this section with one more lemma.

LEMMA. *If μ is a probability measure on X for which T is measure-preserving, then μ is of the form μ_a for some $\boldsymbol{a}$ satisfying* (1) *and* (2).

Proof. We must set $a_n = \mu(n)$, where (n) is the rectangle consisting of all sequences in X beginning with n. Given any rectangle $(r_1, r_2, \cdots, r_k)$, we may write it as the image, under an appropriate power of T^{-1}, of the rectangle

$$(\underbrace{r_k, r_k, \cdots, r_k}_{k}).$$

Hence

$$\mu(r_1, r_2, \cdots, r_k) = \mu(\underbrace{r_k, r_k, \cdots, r_k}_{k}).$$

Now by induction on n we can prove that

$$(\Delta^n \boldsymbol{a})_k = \mu(\underbrace{k, k, \cdots, k}_{n+1}) \quad (k = 0, 1, 2, \cdots). \tag{8}$$

For $n = 0$, this is just the definition of a_k. Suppose (8) is true for $n = m - 1$. Then

$$\begin{aligned}
(\Delta^m \boldsymbol{a})_k &= (\Delta^{m-1}\boldsymbol{a})_k - (\Delta^{m-1}\boldsymbol{a})_{k+1} \\
&= \mu(\underbrace{k, k, \cdots, k}_{m}) - \mu(\underbrace{k+1, k+1, \cdots, k+1}_{m}) \\
&= \sum_{i=k}^{\infty} \mu(\underbrace{k, k, \cdots, k}_{m}, i) - \mu(\underbrace{k+1, k+1, \cdots, k+1}_{m}) \\
&= \sum_{i=k+1}^{\infty} \mu(\underbrace{k, k, \cdots, k}_{m}, i) - \mu(\underbrace{k+1, k+1, \cdots, k+1}_{m}) + (\underbrace{k, k, \cdots, k}_{m+1}) \\
&= \sum_{i=k+1}^{\infty} \mu(\underbrace{k+1, k+1, \cdots, k+1}_{m}, i) - \mu(\underbrace{k+1, k+1, \cdots, k+1}_{m}) \\
&\qquad + \mu(\underbrace{k, k, \cdots, k}_{m+1}) = \mu(\underbrace{k, k, \cdots, k}_{m+1}).
\end{aligned}$$

This proves (8) for all n and hence proves the lemma.

The decomposition of μ. Let us first discuss the continuity properties of the transformation T.

LEMMA. *The transformation T on the set X is continuous. Moreover, T restricted to the subset U of X consisting of unbounded points*

$$U = \{\omega \in X : \lim_n \omega_n = \infty\}$$

is one-to-one and onto.

Proof. The second statement of the lemma is clear. To prove the first, suppose ρ is a point in X and that ρ^n is a sequence of points in X converging to ρ. If ρ is not a constant sequence, $T\rho$ is determined by a finite number k of coordinates of ρ, say $\rho_1, \cdots, \rho_k$. But for large n the first k coordinates of ρ^n must also be $\rho_1, \cdots, \rho_k$, and hence $T(\rho^n)$ converges to $T\rho$. On the other hand, if ρ is a constant sequence, the first l coordinates of ρ^n are eventually all the same so that $T(\rho^n)$ converges to $(0,0,\cdots)$ which is of course $T\rho$.

Thus we can say that the system (T, U) is a *Borel system*. As Oxtoby has shown, the essential features of the Kryloff-Bogoliouboff decomposition remain valid in such a system. Let us emphasize the fact that the results of the Kryloff-Bogoliouboff paper do not apply to our situation directly, although perhaps through a modification of the space X (or of its topology) they could be made applicable.

The result that we make use of is this: to almost every point ω of X (i.e., to every point in a set which has measure one under *any* invariant measure) we can assign an ergodic measure μ_ω in such a way that, for any bounded Borel-measurable function f on X, the function

$$\int f d\mu_\omega \tag{9}$$

is measurable and, moreover, for any invariant measure μ_a on X, we have

$$\mu_a(R) = \int_X \mu_\omega(R) d\mu_a(\omega) \tag{10}$$

for any rectangle R in X.

Since μ_ω is ergodic, we have seen (7) that $\mu_\omega = \mu_\theta$ for some θ between 0 and 1. Let this number be $\theta(\omega)$. If, in equation (9), we put for f the characteristic function of the rectangle (0), we see that

$$\theta(\omega) = 1 - \int_X f d\mu_\omega$$

is a measurable function on X. If we let R be the rectangle (n), equation (10) yields

$$a_n = \int_X \mu_{\theta(\omega)}(n) d\mu_a(\omega) = \int_X \theta(\omega)^n (1 - \theta(\omega)) d\mu_a(\omega). \tag{11}$$

Let $G_a(x)$ be the distribution function for $\theta(\omega)$ with respect to the measure μ_a, i.e.,

$$G_a(x) = \mu_a\{\theta(\omega) \leqq x\}$$

so that (11) becomes

$$a_n = \int_0^1 t^n (1 - t) dG_a(t) \quad (n = 0, 1, 2, \cdots).$$

Setting $F_a(x) = \int_0^x (1-t)\, dG_a(t)$ we get the usual form of the representation:

$$a_n = \int_0^1 t^n dF_a(t) \quad (n = 0, 1, 2, \cdots).$$

We are now in a position to prove the following theorem.

THEOREM. *If* $\boldsymbol{b}=(b_0,b_1,b_2,\cdots)$ *is any completely monotone sequence, there is an increasing function* $F_b(x)$ *defined on the unit interval such that*

$$b_n = \int_0^1 t^n dF_b(t) \quad (n=0,1,2,\cdots). \tag{12}$$

Proof. Let $b_* = \lim b_n$. The sequence $\boldsymbol{a}$ defined by $a_n = (b_n - b_{n+1})/(b_0 - b_*)$ $(n=0,1,2,\cdots)$ is of course completely monotone, but also satisfies (2) (unless $b_0 = b_1 = \cdots = b_*$, but this case can easily be handled separately). Hence, from the remarks above, we have $a_n = \int_0^1 t^n dF_a(t)$ $(n=0,1,2,\cdots)$. Let $F_b(x) = (b_0 - b_*)L(x) + b_*\delta(x)$ where $L(x) = \int_0^x (1/(1-t))dF_a(t)$ and $\delta(x)=1$ if $x=1$ and zero otherwise. Then

$$\begin{aligned}
\int_0^1 t^n dF_b(t) &= (b_0 - b_*)\int_0^1 (t^n/(1-t))dF_a(t) + b_*\int_0^1 t^n d\delta(t)\\
&= (b_0 - b_*)\left(\int_0^1 (1/(1-t))dF_a(t) - \sum_{k=0}^{n-1}\int_0^1 t^k dF_a(t)\right) + b_*\\
&= b_* + (b_0 - b_*)\left(\int_0^1 (1/(1-t))dF_a(t) - \sum_{k=0}^{n-1}(b_k - b_{k+1})/(b_0 - b_*)\right)\\
&= b_* + b_n - b_0 + (b_0 - b_*)\int_0^1 (1/(1-t))dF_a(t)\\
&= b_* + b_n - b_0 + (b_0 - b_*)\int_0^1 dG_a(t)\\
&= b_n.
\end{aligned}$$

Hence (12) holds and this proves the theorem.

An inversion formula. We will now try to express the function G_a explicitly in terms of the sequence $\boldsymbol{a}$.

LEMMA. *There is a subset* V *of* X *which has measure one under any* T-*invariant probability measure and is such that if* ω *is any point of* V,

$$\lim_n (\omega_n/n) = \theta(\omega)/(1-\theta(\omega)). \tag{13}$$

Proof. We will represent the space X in yet another way as the space Y of *all* sequences of non-negative integers. We map a point ω in X to the sequence $(\omega_1, \omega_2, -\omega_1, \omega_3 - \omega_2, \cdots)$ in Y. If μ_θ is an ergodic measure on X, we find that the corresponding measure ν_θ on Y is given by

$$\nu_\theta(s_1, s_2, \cdots, s_k) = \prod_{i=1}^{k} (1-\theta)\theta^{s_i}$$

for any rectangle $S = (s_1, s_2, \cdots, s_k)$ in Y. Hence ν_θ is a product measure in Y which is invariant under the shift transformation $(s_1, s_2, s_3, \cdots) \to (s_2, s_3, s_4, \cdots)$. We apply the ergodic theorem, for the shift transformation, to the function f defined on Y by $f(s_1, s_2, \cdots) = s_1$. Then for almost every (ν_θ) point $(s_1, s_2, \cdots)$ in Y we have

$$\lim_n (s_1 + s_2 + \cdots + s_n)/n = \int_Y f(s)d\nu_\theta(s)$$
$$= \sum_{i=0}^{\infty} i(1-\theta)\theta^i = \theta/(1-\theta)$$

or, in X,

$$\lim_n (\omega_n / n) = \theta/(1-\theta) \quad \mu_\theta\text{-a.e.}$$

Clearly $\theta(\omega) = \theta$ on a set V_θ of μ_θ-measure one. Hence the set V on which the two measurable functions $\theta(\omega)/(1-\theta(\omega))$ and $\lim(\omega_n/n)$ coincide is a set which has measure one for any T-invariant measure. This proves the lemma.

THEOREM. *If* $\boldsymbol{a}$ *is any completely monotone sequence satisfying* (2), G_a *is given by*

$$G_a(x) = \lim_n \sum_{i=0}^{[nx/(1-x)]} \binom{n+i-1}{n-1} (\Delta^n \boldsymbol{a})_i \tag{14}$$

at every point x *at which* G_a *is continuous.*

Proof. First, from (13), we see that $\theta(\omega) = \lim_n \omega_n/(n+\omega_n)$, so that

$$\mu_a[\theta(\omega) < x] \leqq \lim_n \mu_a[\omega_n/(n+\omega_n) < x]$$
$$\leqq \mu_a[\theta(\omega) \leqq x].$$

We calculate the center member of this inequality:

$$\mu_a[\omega_n/(n+\omega_n) < x] = \mu_a[\omega_n < nx/(1-x)]$$
$$= \sum_{i=0}^{[nx/(1-x)]} \mu_a[\omega_n = i] = \sum_{i=0}^{[nx/(1-x)]} \binom{n+i-1}{n-1} (\Delta^n \boldsymbol{a})_i$$

where

$$\binom{n+i-1}{n-1}$$

is the number of rectangles of length n ending in i. Now, passing to the limit, we get

$$G_a(x-) \leqq \sum_{i=0}^{[nx/(1-x)]} \binom{n+i-1}{n-1} (\Delta^n \boldsymbol{a})_i \leqq G_a(x),$$

proving the theorem.

The transformations T_θ. We can obtain some information about the spectra of the measure-preserving transformations T_θ, although the results in this section are incomplete and probably not the best possible. Let us begin with the following theorem.

THEOREM. *If α is an eigenvalue of the transformation T_θ, then*

$$\lim_n \alpha^{B(n)} = 1 \quad \mu_\theta\text{-}a.e.$$

where $B(n)$ is the binomial coefficient

$$\binom{n+\omega_n}{n}.$$

Proof. Let f be the eigenfunction having eigenvalue α. For each ω in X, let ω^n be the rectangle $(\omega_1, \omega_2, \cdots, \omega_n)$ of length n; let $R(\omega,i,j)$ be the rectangle $(\omega_n, \omega_n, \cdots, \omega_n, i, j)$ of length $n+2$, and $S(\omega,i,j)$ be the rectangle $(\omega_1, \omega_2, \cdots, \omega_n, i, j)$ of length $n+2$. It is known that

$$\lim_n (1/\mu_\theta(\omega^n)) \int_{\omega^n} f\,d\mu_\theta = f(\omega) \quad \text{a.e.}$$

and hence

$$\lim_n \left| (1/\mu_\theta(\omega^n)) \int_{\omega^n} f\,d\mu_\theta \right| = 1 \quad \text{a.e.}$$

Estimating the value of $\left| \int_{\omega^n} f\,d\mu_\theta \right|$, we find

$$\left| \int_{\omega^n} f\,d\mu_\theta \right| = \left| \sum_{j \geqq i \geqq \omega_n} \int_{S(\omega,i,j)} f\,d\mu_\theta \right| = \left| \sum_{j \geqq i \geqq \omega_n} \int_{R(\omega,i,j)} f\,d\mu_\theta \right|$$

$$\leqq \sum_{j=\omega_n}^{\infty} \left| \sum_{i=\omega_n}^{\infty} \int_{R(\omega,i,j)} f\,d\mu_\theta \right| \leqq \sum_{j=\omega_n}^{\infty} \theta^j (1-\theta)^{n+2} |G_n(j)|$$

where $G_n(j)$ is a sum of powers of α, the powers being those powers of T which send the rectangle $R(\omega, \omega_n, j)$ into the rectangle $R(\omega,i,j)$ $(i = \omega_n, \omega_n + 1, \cdots, j)$.

Let $S(n,r)$ be the number of rectangles of length n which end in r. Let $P(n,s) = S(n,0) + S(n,1) + \cdots + S(n,s)$. Note that the number $B(n)$, appearing in the statement of the theorem, is $P(n, \omega_n)$.

We leave to the reader the laborious task of showing that

$$\text{(15)} \qquad |G_n(j)| = \left| 1 + \alpha^{P(n\ \omega_n)} + \cdots + \alpha^{P(n\ \omega_n) + \cdots + P(n, j-1)} \right|.$$

Notice that this implies $|G_n(j)| \leqq j - \omega_n + 1$. Now we have

$$\left| (1/\mu(\omega^n)) \int_{\omega^n} f\,d\mu \right| \leqq (1/\theta^{\omega_n}(1-\theta)^n) \sum_{j=\omega_n}^{\infty} \theta^j (1-\theta)^{n+2} |G_n(j)| \leqq 1.$$

Both sides of this inequality approach 1 as n approaches infinity, so, for the middle,

$$\lim_n (1-\theta)^2 \sum_{j=0}^{\infty} \theta^j \mid G_n(\omega_n + j) \mid = 1 \quad \text{a.e.}$$

But this can happen only if $\mid G_n(\omega_n + j) \mid$ approaches $j+1$ almost everywhere. Referring to (15), we see that $\alpha^{B(n)}$ approaches 1, almost everywhere. This is what we wished to prove.

THEOREM. *If $\theta < 1/2$, the transformation T_θ has no prime roots of unity as eigenvalues.*

Proof. Suppose α is a pth root of unity and is an eigenvalue. It is easy to see that the binomial coefficients

$$\binom{p^k + i}{p^k} \quad (k = 1,2,\cdots; i = 0,1,\cdots,p^k - 1)$$

are all equal to 1 (mod p). From the previous theorem we have

$$\lim_k \alpha^{\binom{p^k + \omega_{p^k}}{p^k}} = 1 \quad \text{a.e.} \tag{16}$$

On the other hand, from (13), we have

$$\lim_k \omega_{p^k} / p^k = \theta/(1-\theta) < 1 \quad \text{a.e.}$$

since $\theta < 1/2$. That is, eventually $\omega_{p^k} < p^k$, so that eventually

$$\binom{p^k + \omega_{p^k}}{p^k} = 1 \quad (\text{mod } p). \tag{17}$$

Hence (16) and (17) together form the contradiction which proves the theorem.

We obtain the immediate

COROLLARY. *If $\theta < 1/2$, all powers of T_θ are ergodic.*

It is interesting to notice that the infinite product:

$$f(\omega) = \alpha^{\binom{1+\omega_1-1}{1}} \alpha^{\binom{2+\omega_2-1}{2}} \cdots \alpha^{\binom{k+\omega_k-1}{k}} \cdots$$

if it converges, is an eigenfunction with eigenvalue α. We have seen already that if α is an eigenvalue, than

$$\lim_n \alpha^{\binom{n+\omega_n}{n}} = 1 \quad \text{a.e.} \tag{18}$$

Integration of (18) over X yields

$$\lim_{n} (1-\theta)^n \sum_{j=0}^{\infty} \theta^j \binom{n+j-1}{n-1} \alpha^{\binom{n+j}{n}} = 1.$$

Also left open is the question of the conjugacy of T_θ and T_θ^{-1}, and the question mentioned earlier, of the conjugacy of T_θ and T_ϕ, $\theta \neq \phi$.

Bibliography

1. W. Ambrose and S. Kakutani, *Structure and continuity of measurable flows*, Duke Math. J. **9** (1942), 27–42.
2. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.
3. W. Feller, *Completely monotone functions and sequences*, Duke Math. J. **5** (1939), 661–674.
4. N. Kryloff and N. Bogoliouboff, *La théorie générale de la mesure dans son application à l'étude des systems dynamiques de la mécanique non linéaire*, Ann. of Math. (2) **38** (1937), 65–113.
5. J. C. Oxtoby, *Ergodic sets*, Bull. Amer. Math. Soc. **58** (1952), 116–136.
6. D. V. Widder, *The Laplace transform*, Princeton Univ. Press, Princeton, N. J., 1941.

Duke University,
Durham, North Carolina